The Chern invariants for parabolic bundles at multiple points

Chadi Taher

THE CHERN INVARIANTS FOR PARABOLIC BUNDLES AT MULTIPLE POINTS

CHADI HASSAN TAHER

Abstract. If $D \subset X$ is a curve with multiple points in a surface, a parabolic bundle defined on (X, D) away from the singularities can be extended in several ways to a parabolic bundle on a resolution of singularities. We investigate the possible parabolic Chern classes for these extensions.

1. Introduction

Suppose $\check{X}$ is a smooth surface and $\check{D} = \check{D}_1 + \ldots + \check{D}_k$ is a divisor with each $\check{D}_i$ smooth. Suppose $\check{E}$ is a bundle provided with filtrations $\check{F}^i_\cdot$ along the $\check{D}_i$, and parabolic weights $\alpha^i_\cdot$. If $\check{D}$ has normal crossings, this defines a locally abelian parabolic bundle on $(\check{X}, \check{D})$ and the parabolic Chern classes have been calculated as explained in the previous part.

Suppose that the singularities of $\check{D}$ contain some points of higher multiplicity. For the present work we assume that these are as easy as possible, namely several smooth branches passing through a single point with distinct tangent directions. The first basic case is a triple point.

Let $\varphi : X \to \check{X}$ denote this birational transformation, and let $D_i \subset X$ denote the strict transforms of the $\check{D}_i$. Assuming for simplicity that there is a single multiple point, denote by D_0 the exceptional divisor. Now $D = D_0 + \cdots + D_k$ is a divisor with normal crossings. Suppose E is a vector bundle on X with

$$E|_{X-D_0} = \varphi^*(\check{E})|_{X-D_0}.$$

The filtrations $\check{F}^i_\cdot$ induce filtrations of $\varphi^*(\check{E})|_{D_i}$ and hence of $E|_{D_i - D_i \cap D_0}$, which then extend uniquely to filtrations $F^i_\cdot$ of $E|_{D_i}$. Associate to these filtrations the same parabolic weights as before.

Up until now we have already made a choice of extension of the bundle E. Choose furthermore a filtration $F^0_\cdot$ of $E|_{D_0}$ and parabolic weights associated to D_0. Having made these choices we get a parabolic bundle on the normal crossings divisor (X, D), which determines parabolic Chern classes. We are particularly

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interested in the invariant Δ which combines c_1 and c_2 in such a way as to be invariant by tensoring with a line bundle.

The goal of this paper is to provide a convenient calculation of Δ and then investigate its dependence on the choices which have been made above. In particular we would like to show that Δ achieves its minimum and calculate this minimum, which can be thought of as the Chern invariant associated to the original parabolic structure on the multiple point singularity $(\check{X}, \check{D})$.

The main difficulty is to understand the possible choices for E. For this we use the technique of Ballico-Gasparim [Ba] [BG1] [BG2].

2. Calculating the invariant Δ of a locally abelian parabolic bundle

Recall from [Ta] formulas for the parabolic first, second Chern characters of a locally abelian parabolic bundle E in codimension one and two, $ch_1^{Par}(E)$, and $ch_2^{Par}(E)$.

Let X be a smooth projective variety over an algebraically closed field of characteristic zero and let D be a strict normal crossings divisor on X. Write $D = D_1 + ... + D_n$ where D_i are the irreducible smooth components, meeting transversally. We sometimes denote by $\mathcal{S} := \{1, \ldots, n\}$ the set of indices for components of the divisor D.

For $i = 1, ..., n$, let Σ_i be finite linearly ordered sets with notations $\eta_i \leq ... \leq \sigma \leq \sigma' \leq \sigma'' \leq ... \leq \tau_i$ where η_i is the smallest element of Σ_i and τ_i the greatest element of Σ_i.

Let Σ'_i be the set of connections between the σ's i.e

$$\Sigma'_i = \{(\sigma, \sigma'),\ s.t\ \sigma < \sigma' \ and\ there\ exist\ no\ \sigma''\ with\ \sigma < \sigma'' < \sigma'\}.$$

Consider the *tread functions* $m_+ : \Sigma'_i \to \Sigma_i$ and $m_- : \Sigma'_i \to \Sigma_i$ if $\lambda = (\sigma, \sigma') \in \Sigma'_i$ then $\sigma = m_-(\lambda), \sigma' = m_+(\lambda)$. In the other direction, consider the *riser functions* $C_+ : \Sigma_i - \{\tau_i\} \to \Sigma'_i$ and $C_- : \Sigma_i - \{\eta_i\} \to \Sigma'_i$ such that $C_+(\sigma) = (\sigma, \sigma')$ where $\sigma' > \sigma$ the next element and $C_-(\sigma) = (\sigma'', \sigma)$ where $\sigma'' < \sigma$ the next smaller element.

For any parabolic bundle E in codimension one, and two, the parabolic first, second Chern characters $ch_1^{Par}(E), and\ ch_2^{Par}(E),$ are obtained as follows:

- $$ch_1^{Par}(E) := ch_1^{Vb}(E) - \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \Sigma'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]$$
- $$ch_2^{Par}(E) := ch_2^{Vb}(E) - \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \Sigma'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).(\xi_{i_1})_\star \left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right)$$

$$+\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$+\frac{1}{2}\sum_{i_1\neq i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p].$$

Where:

- $ch^{Vb}_1(E), ch^{Vb}_2(E)$ denotes the first, second, Chern character of vector bundles E.
- $Irr(D_I)$ denotes the set of the irreducible components of $D_I := D_{i_1}\cap D_{i_2}\cap ...\cap D_{i_q}$.
- ξ_I denotes the closed immersion $D_I \longrightarrow X$, and $\xi_{I,\star} : A^k(D_I) \longrightarrow A^{k+q}(X)$ denotes the associated Gysin map.
- Let p be an element of $Irr(D_i\cap D_j)$. Then $rank_p(Gr^I_\lambda)$ denotes the rank of Gr^I_λ as an $\mathcal{O}_p$-module.
- $[D_{i_j}]\in A^1(X)\otimes\mathbb{Q}$, and $[D_p]\in A^2(X)\otimes\mathbb{Q}$ denote the cycle classes given by D_{i_j} and D_p respectively.

Definition 2.1. *Let $Gr^{i_1}_{\lambda_{i_1}}$ be a bundle over D_{i_1}. Define the degree of $Gr^{i_1}_{\lambda_{i_1}}$ to be*

$$\deg(Gr^{i_1}_{\lambda_{i_1}}) := (\xi_{i_1})_\star\left(c_1^{D_{i_1}}(Gr^{i_1}_{\lambda_{i_1}})\right).$$

Definition 2.2. *The invariant Δ, which is a normalized version of c_2 designed to be independent of tensorization by line bundles. It is defined by*

$$\Delta = c_2 - \frac{r-1}{2r}c_1^2.$$

Recall that: $ch_2 = \frac{1}{2}c_1^2 - c_2 \Longrightarrow c_2 = \frac{1}{2}c_1^2 - ch_2$. Therefore

$$\Delta = \frac{1}{2}c_1^2 - ch_2 - \frac{1}{2}c_1^2 + \frac{1}{2}c_1^2 + \frac{1}{2r}c_1^2 = \frac{1}{2r}ch_1^2 - ch_2.$$

Then $\Delta^{Par}(E) = \frac{1}{2r}ch_1^{Par}(E)^2 - ch_2^{Par}(E)$

$$= \frac{1}{2r}\left[ch^{Vb}_1(E) \; - \sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\Sigma'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]\right]^2$$

$$- ch_2^{Vb}(E) \ + \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).deg(Gr^{i_1}_{\lambda_{i_1}})$$

$$- \frac{1}{2} \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} \alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$- \frac{1}{2} \sum_{i_1 \neq i_2} \sum_{\substack{\lambda_{i_1} \\ \lambda_{i_2}}} \sum_{p \in Irr(D_{i_1} \cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p].$$

$$= \frac{1}{2r} ch_1^{Vb}(E)^2$$

$$- \frac{1}{r} \left[ch_1^{Vb}(E) \right] \cdot \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \Sigma'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]$$

$$+ \frac{1}{2r} \sum_{i_1 \neq i_2} \sum_{\substack{\lambda_{i_1} \\ \lambda_{i_2}}} \sum_{p \in Irr(D_{i_1} \cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank(Gr^{i_1}_{\lambda_{i_1}})rank(Gr^{i_2}_{\lambda_{i_2}}).[D_p].$$

$$+ \frac{1}{2r} \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \Sigma'_{i_1}} \sum_{\lambda'_{i_1} \in \Sigma'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_1}(\lambda'_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).rank(Gr^{i_1}_{\lambda'_{i_1}}).[D_{i_1}]^2$$

$$- ch_2^{Vb}(E) \ + \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).deg(Gr^{i_1}_{\lambda_{i_1}})$$

$$- \frac{1}{2} \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \sum'_{i_1}} \alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$- \frac{1}{2} \sum_{i_1 \neq i_2} \sum_{\substack{\lambda_{i_1} \\ \lambda_{i_2}}} \sum_{p \in Irr(D_{i_1} \cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_p(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_p].$$

Proposition 2.3. $\Delta^{Par}(E) = \Delta^{Vb}(E)$

$$- \frac{1}{r} ch_1^{Vb}(E) \cdot \sum_{i_1 \in \mathcal{S}} \sum_{\lambda_{i_1} \in \Sigma'_{i_1}} \alpha_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]$$

$$+\frac{1}{2r}\sum_{i_1\neq i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{p\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank(Gr^{i_1}_{\lambda_{i_1}})rank(Gr^{i_2}_{\lambda_{i_2}})[D_p].$$

$$+\frac{1}{2r}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\Sigma'_{i_1}}\sum_{\lambda'_{i_1}\in\Sigma'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_1}(\lambda'_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).rank(Gr^{i_1}_{\lambda'_{i_1}}).[D_{i_1}]^2$$

$$+\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha_{i_1}(\lambda_{i_1}).\deg(Gr^{i_1}_{\lambda_{i_1}})$$

$$-\frac{1}{2}\sum_{i_1\in\mathcal{S}}\sum_{\lambda_{i_1}\in\sum'_{i_1}}\alpha^2_{i_1}(\lambda_{i_1}).rank(Gr^{i_1}_{\lambda_{i_1}}).[D_{i_1}]^2$$

$$-\frac{1}{2}\sum_{i_1\neq i_2}\sum_{\substack{\lambda_{i_1}\\ \lambda_{i_2}}}\sum_{y\in Irr(D_{i_1}\cap D_{i_2})}\alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_y(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[D_y].$$

3. Parabolic bundles with full flags

We use the fact that X is a surface to simplify the above expressions, by assuming that the parabolic filtrations are full flags.

Proposition 3.1. *If E' is a locally free sheaf over $X-\{P\}$, then $\exists!$ extension to a locally free sheaf E over X s.t $E'|_{X-\{P\}} = E$.*

Proposition 3.2. *If we have a strict sub-bundle of $E|D_i-\{P\}$ then $\exists!$ extension to a strict sub-bundle of $E|D_i$.*

Remark 1. *It follows from these propositions that if $(E', F'^i_{\alpha_i})$ is a parabolic structure over $(X-\{P\}, D-\{P\})$, then we obtain a bundle E over X with the filtrations $\{F'^i_{\alpha_i}\}$ of $E|_{D_i}$ by a strict sub-bundles.*

Definition 3.3. $Hom(\mathcal{O}_{\mathbb{P}^1}(-m_i), \mathcal{O}^{\oplus 2}_{\mathbb{P}^1}) = H^0(\mathbb{P}^1, \mathcal{O}(-m_i)\otimes\mathcal{O})^{\oplus 2} = H^0(\mathbb{P}^1, \mathcal{O}(m_i))^{\oplus 2}$.

For example, subbundles of a rank two trivial bundle may be expressed very explicitly.

Proposition 3.4. *Consider the two polynomials $(A_i, B_i) \in H^0(\mathbb{P}^1, \mathcal{O}(m_i))^{\oplus 2}$, the sub-sheaves are saturated iff $m_i = ((max(deg(A_i), deg(B_i))$ and $(A_i, B_i) = 1$. Then there is an isomorphism*

$$(A_i, B_i) : \mathcal{O}_{\mathbb{P}^1}(-m_i) \longrightarrow \mathcal{O}^{\oplus 2}_{\mathbb{P}^1}$$

Lemma 3.5. *$\forall\ 0 \subseteq F^i_{\sigma_i} \subseteq F^i_{\sigma'_i} \subseteq .... \subseteq F^i_{\tau_i} \subseteq E|_{D_i}$, $\exists$ complet flags $0 \subseteq \widehat{F}_1 \subseteq ... \subseteq \widehat{F}_r = E|_{D_i}$ s.t $F^i_{\sigma_i} = \widehat{F}_{k(\sigma_i)}$. where $\forall\ \sigma_i \in \Sigma_i$ we have $k(i) \in \{0, 1, ..., r\}$ then $\exists$ $k : \Sigma_i \longrightarrow \{0, 1, ..., r\}$ s.t $k = rank(F^i_{\sigma_i})$.*

In view of this lemma, we will now suppose that all the filtrations are complete flags. The weights should then form an increasing sequence but not necessarily strictly increasing.

In particular we will change notation and denote the filtration of $E|_{D_i}$ by

$$0 = F_0^i \subseteq F_1^i \subseteq .... \subseteq F_r^i = E|_{D_i}.$$

In this case $\Sigma_i = \{\sigma_i^0, \dots, \sigma_i^r\}$ and $\Sigma_i' = \{\lambda_i^1, \dots, \lambda_i^r\}$. These sets now have the same number of elements for each i so we can return to a numerical indexation. We denote

$$Gr_k^i(E|_{D_i}) := Gr^i_{\lambda_i^k}(E|_{D_i}) = F_k^i / F_{k-1}^i.$$

Proposition 3.6. *$rank(Gr^i_{\lambda_i}) = 1 \iff$ the filtrations $F^i_{\sigma_i^1} \le F^i_{\sigma_i^2} \le ... \le F^i_{\sigma_i^r}$ are complet flags for $i = 1, 2, ..., n$.*

Since we are on a surface, $D_i \cap D_j$ is a finite collection of points. At each point $P \in D_i \cap D_j$ we have two filtrations of E_P coming from the parabolic filtrations along D_i and D_j. We are now assuming that they are both complete flags. The incidence relationship between these filtrations is therefore encoded by a permutation.

Lemma 3.7. *$\forall\, k\ \exists!\, k' \in \{1, ..., r\}$ s.t $rank\left(Gr_k^i Gr_{k'}^i(E_P)\right) = \dfrac{F_k^i \cap F_{k'}^j}{F_{k-1}^j \cap F_{k'}^j + F_k^i \cap F_{k'-1}^i} = 1$.*

Definition 3.8. *$\forall\ P \in D_i \cap D_j$ define the permutation $\sigma(P, i, j) : \{1, ..., r\} = \Sigma_i' \longrightarrow \{1, ..., r\}$ which sends $k \in \{1, ..., r\}$ to $\sigma(P, i, j)(k) = k'$ where k' is the unique index given in the previous lemma.*

Lemma 3.9. *$\forall\, k$ if $k'' \neq \sigma(P, i, j)(k)$ then $rank\left(Gr_k^i Gr_{k''}^i(E_P)\right) = 0$.*

Since the filtrations are full flags, there are r different indices $\lambda_i^1, \dots, \lambda_i^r$ for each divisor D_i. We introduce the notation $\alpha(D_i, k) := \alpha_i(\lambda_i^k)$.

With this notation we obtain the following expression for the term involving $Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}$:

$$-\frac{1}{2} \sum_{i_1 \neq i_2} \sum_{\substack{\lambda_{i_1} \\ \lambda_{i_2}}} \sum_{p \in Irr(D_{i_1} \cap D_{i_2})} \alpha_{i_1}(\lambda_{i_1}).\alpha_{i_2}(\lambda_{i_2}).rank_y(Gr^{i_1,i_2}_{\lambda_{i_1},\lambda_{i_2}}).[y].$$

$$= \qquad -\frac{1}{2} \sum_{i \neq j} \sum_{k=1}^{r} \sum_{y \in Irr(D_i \cap D_j)} \alpha(D_i, k).\alpha(D_j, \sigma(y, i, j)(k)).[y].$$

On the other hand, all ranks of the graded pieces $Gr(D_i, k) := Gr^i_{\lambda_i^k}$ are equal to 1. They are line bundles on D_i.

Definition 3.10. *Suppose $i \neq 0$. Let $Gr(D_i, k)$ are line bundles over D_i. Then we define the $deg Gr(D_i, k)$ to be:*

$$\deg\left(Gr(D_i,k)\right) = (\xi_i)_\star \left(c_1^{D_i}(Gr(D_i,k))\right).$$

We can now rewrite the statement of Proposition.

Proposition 3.11. $\Delta^{Par}(E) = \Delta^{Vb}(E)$

$$\begin{aligned}
&- \frac{1}{r} ch_1^{Vb}(E) \cdot \sum_{i\in\mathcal{S}} \sum_{k=1}^{r} \alpha(D_i,k).[D_i] \\
&+ \frac{1}{2r} \sum_{i\neq j} \sum_{k,l\in[1,r]} \sum_{y\in Irr(D_i\cap D_j)} \alpha(D_i,k).\alpha(D_j,l)[y]. \\
&+ \frac{1}{2r} \sum_{i\in\mathcal{S}} \sum_{k,l\in[1,r]} \alpha(D_i,k).\alpha(D_i,l).[D_i]^2 \\
&+ \sum_{i\in\mathcal{S}} \sum_{k=1}^{r} \alpha(D_i,k).\deg\left(Gr(D_i,k)\right) \\
&- \frac{1}{2} \sum_{i\in\mathcal{S}} \sum_{k=1}^{r} \alpha(D_i,k)^2.[D_i]^2 \\
&- \frac{1}{2} \sum_{i\neq j} \sum_{k=1}^{r} \sum_{y\in Irr(D_i\cap D_j)} \alpha(D_i,k).\alpha(D_j,\sigma(y,i,j)(k)).[y].
\end{aligned}$$

We have $\alpha(D_i,k) \in [-1,0]$, define $\alpha^{tot}(D_i) := \sum_{k=1}^{r} \alpha(D_i,k)$. With this notation,
$\Delta^{Par}(E) = \Delta^{Vb}(E)$

$$\begin{aligned}
&- \frac{1}{r} ch_1^{Vb}(E) \cdot \sum_{i\in\mathcal{S}} \alpha^{tot}(D_i)[D_i] \\
&+ \frac{1}{2r} \sum_{i\neq j} \alpha^{tot}(D_i)\alpha^{tot}(D_j)[D_i\cap D_j] \\
&+ \frac{1}{2r} \sum_{i\in\mathcal{S}} \alpha^{tot}(D_i)^2.[D_i]^2
\end{aligned}$$

$$+ \sum_{i\in\mathcal{S}}\sum_{k=1}^{r}\alpha(D_i,k).$$

$$- \frac{1}{2}\ \sum_{i\in\mathcal{S}}\sum_{k=1}^{r}\alpha(D_i,k)^2.[D_i]^2$$

$$- \frac{1}{2}\sum_{i\neq j}\sum_{k=1}^{r}\ \sum_{y\in Irr(D_i\cap D_j)}\alpha(D_i,k).\alpha(D_j,\sigma(y,i,j)(k)).[y].$$

For more simplification of $\Delta^{Par}(E)$, define β such that

$$\beta(D_i,k) := \alpha(D_i,k) - \frac{\alpha^{tot}(D_i)}{r} \implies \alpha(D_i,k) = \beta(D_i,k) + \frac{\alpha^{tot}(D_i)}{r}.$$

Remark 2. *We remark that* $\sum_{k=1}^{r}\beta(D_i,k) = 0$*. Hence*

$$\sum_{k=1}^{r}\alpha(D_i,k)^2 = \sum_{k=1}^{r}\beta(D_i,k)^2 + \frac{\alpha^{tot}(D_i)^2}{r},$$

and for $i\neq j$ *and* $y\in Irr(D_i\cap D_j)$*,*

$$\sum_{k=1}^{r}\alpha(D_i,k).\alpha(D_j,\sigma(y,i,j)(k)) = \sum_{k=1}^{r}\alpha(D_i,k).\alpha(D_j,\sigma(y,i,j)(k)) + \frac{\alpha^{tot}(D_i).\alpha^{tot}(D_j)}{r}.$$

Furthermore note that

$$\sum_{k=1}^{r}(\xi_i)_\star\left(c_1^{D_i}(Gr(D_i,k))\right) = (\xi_i)_\star\left(c_1^{D_i}(E|_{D_i})\right) = c_1^{Vb}(E).[D_i]$$

so

$$\sum_{k=1}^{r}\alpha(D_i,k).\deg\left(Gr(D_i,k)\right) = \sum_{k=1}^{r}\beta(D_i,k).\deg\left(Gr(D_i,k)\right) + \frac{\alpha^{tot}(D_i)c_1^{Vb}(E).[D_i]}{r}.$$

Using these remarks and the previous formula we get

$$\Delta^{Par}(E) = \Delta^{Vb}(E)$$

$$- \frac{1}{r}ch_1^{Vb}(E)\cdot\sum_{i\in\mathcal{S}}\alpha^{tot}(D_i)[D_i]$$

$$+\ \frac{1}{2r}\sum_{i\neq j}\alpha^{tot}(D_i)\alpha^{tot}(D_j)[D_i\cap D_j]$$

$$+\ \frac{1}{2r}\sum_{i\in\mathcal{S}}\alpha^{tot}(D_i)^2.[D_i]^2$$

$$+\ \sum_{i\in\mathcal{S}}\sum_{k=1}^{r}\beta(D_i,k).deg(Gr(D_i,k))$$

$$+\ \frac{1}{r}\sum_{i\in\mathcal{S}}\alpha^{tot}(D_i).c_1^{Vb}(E).[D_i]$$

$$-\ \frac{1}{2}\ \sum_{i\in\mathcal{S}}\sum_{k=1}^{r}\beta(D_i,k)^2.[D_i]^2$$

$$-\ \frac{1}{2r}\ \sum_{i\in\mathcal{S}}\alpha^{tot}(D_i)^2.[D_i]^2$$

$$-\ \frac{1}{2}\sum_{i\neq j}\sum_{k=1}^{r}\sum_{y\in Irr(D_i\cap D_j)}\beta(D_i,k).\beta(D_j,\sigma(y,i,j)(k)).[y].$$

$$-\ \frac{1}{2r}\sum_{i\neq j}\sum_{k=1}^{r}\sum_{y\in Irr(D_i\cap D_j)}\alpha^{tot}(D_i).\alpha^{tot}(D_j).[y].$$

The terms containing $\alpha^{tot}(D_i)$ all cancel out, giving the following formula.

Proposition 3.12. $\Delta^{Par}(E)=\Delta^{Vb}(E)$

$$+\ \sum_{i\in\mathcal{S}}\sum_{k=1}^{r}\beta(D_i,k).\deg\left(Gr(D_i,k)\right)$$

$$-\ \frac{1}{2}\ \sum_{i\in\mathcal{S}}\sum_{k=1}^{r}\beta(D_i,k)^2.[D_i]^2$$

$$-\ \frac{1}{2}\sum_{i\neq j}\sum_{k=1}^{r}\sum_{y\in Irr(D_i\cap D_j)}\beta(D_i,k).\beta(D_j,\sigma(y,i,j)(k)).[y].$$

The fact that $\Delta^{Par}(E)$ is independent of α^{tot} is the parabolic version of the invariance of Δ under tensoring with line bundles. Even though this is the theoretical explanation, for the proof it was more convenient to calculate explicitly the formula and notice that the terms containing α^{tot} cancel out, than to try to compute the tensor product with a parabolic line bundle.

4. Resolution of singular divisors

Now we can consider a more general situation, where $\check{X}$ is a smooth projective surface but $D = \bigcup_{i=1}^{n} D_i$ is a divisor which may have singularities worse than normal crossings. Let $\check{P} = \{\check{P}_1, ..., \check{P}_r\}$ be a set of points. Assume that the points $\check{P}_j$ are crossing points of $\check{D}_i$, and that they are general multiple points, that is through a crossing point P_j we have divisors $\check{D}_{i_1}, ..., \check{D}_{i_m}$ which are pairwise transverse. Assume that $\check{D}$ has normal crossings outside of the set of points $\check{P}$. We choose an embedded resolution given by a sequence of blowing-ups $\varphi : X \to \check{X}$ in r points $\check{P}_1, ..., \check{P}_r$ and P be the exceptional divisor on X, note that P is a sum of disjoint exceptional components $P_i = \varphi^{-1}(\check{P}_i)$ over the points $\check{P}_i$ respectively. The pullback divisor may be written as $D = D_1 + \cdots + D_a + P_1 + \cdots + P_b$ where D_i is the strict transform of a component $\check{D}_i$ of the original divisor, and P_j are the exceptional divisors.

Definition 4.1. *Let E be a bundle over X, and consider the inclusion $i : U \hookrightarrow X$ where $U = X - \bigcup_{i=1}^{k} P_i$ be a smooth connected quasi-projective surface. Hence $P_i = \mathbb{P}^1$ and let the blowing-up $\varphi : X \longrightarrow \check{X}$. Define $\check{E}$ as a unique bundle over $\check{X}$ such that*

$$\check{E}|_U \cong E|_U,$$
$$\check{E} \ is \ locally \ free.$$

This construction allows us to localize the contributions of the Chern classes of E along the exceptional divisors, by comparison with $\varphi^*(\check{E})$.

Definition 4.2. *Let E be a bundle over X. Consider the inclusions $\varphi^\star \check{E} \hookrightarrow i_\star(E|_U)$, where $i_\star(E|_U)$ is a quasi-coherent sheaves over X, and $E \hookrightarrow i_\star(E|_U)$, where $i : U \hookrightarrow \check{X}$. Define E'' to be the intersection of subsheaves $\varphi^\star \check{E}$ and E of $i_\star(E|_U)$.*

Lemma 4.3. *E'' is a free locally coherent sheaf.*

Definition 4.4. *Consider the two exact sequences*

$$0 \longrightarrow E'' \longrightarrow \varphi^\star \check{E} \longrightarrow Q' \longrightarrow 0$$
$$0 \longrightarrow E'' \longrightarrow E \longrightarrow Q \longrightarrow 0$$

Let $E/E'' = Q = \bigoplus_{i=1}^{k} Q_i$ *and* $\varphi^\star \check{E}/E'' = Q' = \bigoplus_{i=1}^{k} Q'_i$. *Define the local contribution to be,*

$$ch^{Vb}(E,P)_{loc} := ch^{Vb}(Q) - ch^{Vb}(Q')$$

Proposition 4.5. *If* $\varphi : \check{X} \longrightarrow X$. *Let* P_i *the blowing-up of* $\check{P}_i$ *where* P_i *is the exceptional divisor for* $i = 1, 2, ...k$. *Then*

$$ch^{Vb}(E) = ch^{Vb}(\varphi^\star(\check{E})) + \sum_{i=1}^{k} ch^{Vb}(E, P_i)_{loc}$$

We have $ch_1^{Vb}(E) \in A^1(X)$. Let $\varphi^\star : A^1(\check{X}) \longrightarrow A^1(X)$; where

$A^1(X) = A^1(\check{X}) \oplus \bigoplus_{i=1}^{k} .\mathbb{Z}.[P_i]$. We have

$P_i.\varphi^\star(\check{D}) = 0$ if $\check{D} \in A^1(\check{X})$ and $P_i.P_j = 0$ if $i \neq j$. Then

$ch_1^{Vb}(E) = \varphi^\star ch_1^{Vb}(\check{E}) + \sum_{i=1}^{k} a_i[P_i] = \varphi^\star ch_1^{Vb}(\check{E}) + \sum_{i=1}^{k} ch_1^{Vb}(E, P_i)_{loc}$.

When we take the square, the cross-terms are zero, indeed $ch_1^{Vb}(E, P_i)_{loc}$ is a multiple of the divisor class $[P_i]$ but $[P_i].[P_j] = 0$ for $i \neq j$, and $[P_i].\varphi^*[C] = 0$ for any divisor C on $\check{X}$. Therefore,

$ch_1^{Vb}(E)^2 = \varphi^\star ch_1^{Vb}(\check{E})^2 + \sum_{i=1}^{k} a_i^2[P_i]^2 = \varphi^\star ch_1^{Vb}(\check{E})^2 + \sum_{i=1}^{k} ch_1^{Vb}(E, P_i)_{loc}^2$.

Lemma 4.6. *If* L *is a line bundle over* X, *then*

$$\Delta(E \otimes L) = \Delta(E).$$

5. Local Bogomolov-Gieseker inequality

The classical *Bogomolov-Gieseker inequality* states that if X is projective and E is a semistable vector bundle then $\Delta(E) \geq 0$. We will see that a local version holds; the first observation is that the invariant Δ can be localized, even though it involves a quadratic term in ch_1.

Definition 5.1.

$$\Delta^{Vb}(E, P_i)_{loc} := \frac{1}{2r} ch_1^{Vb}(E, P_i)_{loc}^2 - ch_2^{Vb}(E, P_i)_{loc}$$

Lemma 5.2. *If* $L = \varphi^\star \check{L}(\sum b_i P_j)$ *is a line bundle over* X. *Then*

$$\Delta^{Vb}(E \otimes L; P_i)_{loc} = \Delta^{Vb}(E, P_i)_{loc}$$

Proposition 5.3.

$$\Delta^{Vb}(E) = \varphi^\star \Delta^{Vb}(\check{E}) + \sum_{i=1}^{k} \Delta^{Vb}(E, P_i)_{loc}$$

In order to get a bound, the technique is to apply the Grothendieck decomposition to analyse more closely the structure of E near the exceptional divisors P_i, following Ballico [Ba] and Ballico-Gasparim [BG1] [BG2] and others.

Theorem 5.4. *Every vector bundle E on $\mathbf{P}^1$ is of the form $\mathcal{O}(m_1)^{r_1} \oplus \cdots \oplus \mathcal{O}(m_a)^{r_a} = \bigoplus_{j=1}^{a} \mathcal{O}(m_j)^{r_j}$, $m_1 < \ldots < m_r$ where $m_j \in \mathbb{Z}$, and the r_j are positive integers with $r_1 + \ldots + r_a = r$. This called the Grothendick decomposition and it is unique.*

Apply this decomposition to the restriction of the bundle E to each exceptional divisor $P_i \cong \mathbf{P}^1$. Thus

$$E|_{P_i} = \mathcal{O}(m_{i,1})^{r_{i,1}} \oplus ... \oplus \mathcal{O}(m_{i,a_i})^{r_{i,a_i}} = \bigoplus_{j=1}^{a_i} \mathcal{O}(m_{i,j})^{r_{i,j}}$$

with $m_{i,1} < \ldots < m_{i,a_i}$.

Proposition 5.5. *Let E be a bundle over X, we have,*

$$m_{i,j} = 0 \iff E \cong \varphi^\star \check{E} \ ,$$

if $E' = E(\sum_i k_i.P_i)$ then $m'_{i,j} = m_{i,j} - k_i$, therefore

$$m_{i,j} = k_i \iff E \cong (\varphi^\star \check{E})(-\sum_i k_i.P_i).$$

In this case we say that E is pure, *it is equivalent to saying that $a_i = 1$.*

See Ballico-Gasparim [BG1].

Definition 5.6. *Let E be a non trivial bundle, and $E|_P = \mathcal{O}(m_1)^{r_1} \oplus ... \oplus \mathcal{O}(m_a)^{r_a}$ be the restriction of the bundle E, for $m_1 < m_2 < ... < m_a$. We define*

$$min(E|_P) := m_1 \ , max(E|_P) := m_a, \text{ and } \varphi(E) = max(E|_P) - min(E|_P).$$

Remark 3. *If $\mu(E) = m_a - m_1 = 0$. Then $E|_P = \mathcal{O}_{\mathbb{P}^1}(m_1)^r$; $E = E^\vee(-m.P)$*

Lemma 5.7. *If we have an exact sequence of bundles over $\mathbb{P}^1$*

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0$$

then

$$min(V) \geq min(min(U), min(W)),$$
$$max(V) \leq max(max(U), max(W)).$$

Proof. Define

$$max(U) = max\{n;\ s.t\ \exists\ \mathcal{O}_{\mathbb{P}^1}(n) \to U\, nontrivial\} = max\{n;\ s.t\ H^0(U(-n)) \neq 0\}$$

$$min(U) = min\{n;\ s.t\ \exists\ U \to \mathcal{O}_{\mathbb{P}^n}(n)\, nontrivial\} = max\{n;\ s.t\ H^0(U^*(n)) \neq 0\}$$

then

$$max(U) \leq max(max(U), max(W))$$

$$min(V) \geq min(min(U), min(W))$$

□

Now we concentrate on one of the exceptional divisors P_i and supress the index i from the notation.

Now for $1 \leq t \leq r$, suppose that $E|_P$ is not pure, and consider the exact sequence

$$\begin{array}{ccc} 0 & & \\ \uparrow & & \\ Q & := & \mathcal{O}(m_1)^{r_1} \\ \uparrow & & \\ E|_{P_i} & := & \mathcal{O}(m_1)^{r_1} \oplus \mathcal{O}(m_2)^{r_2} \oplus \cdots \oplus \mathcal{O}(m_a)^{r_a} \\ \uparrow & & \\ K & := & \mathcal{O}(m_2)^{r_2} \oplus \cdots \oplus \mathcal{O}(m_a)^{r_a} \\ \uparrow & & \\ 0 & & \end{array}$$

Definition 5.8. *Suppose X and D are smooth with $D \overset{i_*}{\hookrightarrow} X$. Let E be a free locally bundle over X. Suppose we have an exact sequence*

$$0 \longrightarrow K \longrightarrow E|_D \longrightarrow Q \longrightarrow 0$$

where $Q = \mathcal{O}(m_1)^{r_1}$ is called constant stabilizer. Define E' to be the elementary transformation of E by

$$E' := Ker(E \to i_*Q).$$

Then the sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow i_*Q \longrightarrow 0.$$

is exact.

Lemma 5.9. *We have an exact sequence*

$$0 \longrightarrow Q(-D) \longrightarrow E' \mid_P \longrightarrow K \longrightarrow 0.$$

Then

$$E'(U) = \{S \in E(U)\, s.t\, S|_{(D\cap U)} \in K(D \cap U)\}.$$

Lemma 5.10. $\mu(E') \leq \mu(E) - 1$ *(if $\mu(E) \geq 1$).*

Proof. We have

$$\mathcal{O}_P(-P) = \mathcal{O}_P(i)$$

apply the Lemma 5.9 we get

$$0 \longrightarrow \mathcal{O}(m_1+1)^{r_1} \longrightarrow E'|_P \longrightarrow \bigoplus_{i=2}^{a} \mathcal{O}(m_i)^{r_i} \longrightarrow 0$$

apply Lemma 5.7, take $min = max = m_1 + 1$ and $min = m_2$ then $max \geq m_1 + 1$ $\Longrightarrow min(E' \mid_P) \leq m_a$. Therefore

$$\mu(E') \leq \mu(E) - 1.$$

□

Lemma 5.11.

$$ch\,(\mathcal{O}_P(m_1)) = (P + (m_1 + \frac{1}{2})).$$

Proof. We have

$$\mathcal{O}(m_1) = \mathcal{O}(-m_1 P),$$

consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-(m_1+1)P) \longrightarrow \mathcal{O}_X(-m_1 P) \longrightarrow \mathcal{O}_P(-m_1 P) \longrightarrow 0$$

then

$$ch(\mathcal{O}_P(-m_1 P)) = e^{-m_1 P} - e^{(-m_1+1)P} = (1 - m_1 P + \frac{m_1^2}{2}P^2) - (P - m_1 P^2 - \frac{P^2}{2})$$
$= (P - (m_1 + \frac{1}{2})P^2)$, but $P^2 = -1$, then $ch(\mathcal{O}_P(-m_1)) = (P + (m_1 + \frac{1}{2}))$. □

Proposition 5.12. *We have*

$$ch_1^{Vb}(E') = ch_1^{Vb}(E) - r_1 P \quad \textit{and} \quad ch_2^{Vb}(E') = ch_2^{Vb}(E) - (m_1 + \frac{1}{2})r_1,$$

then

$$\Delta^{Vb}(E', P) = \frac{1}{2r}(ch_1^{Vb}(E) - r_1 P)^2 - ch_2^{Vb}(E) + (m_1 + \frac{1}{2})r_1,$$

therefore

$$\Delta^{Vb}(E', P) = \Delta^{Vb}(E, P) - \frac{r_1}{r} ch_1^{Vb}(E).P - \frac{r_1^2}{2r} + (m_1 + \frac{1}{2})r_1.$$

We can now calculate using the previous lemma.

$E|_P = \mathcal{O}(m_1)^{r_1} \oplus \mathcal{O}(m_2)^{r_2} \oplus ... \oplus \mathcal{O}(m_k)^{r_k}$, where $\sum_{i=1}^{k} r_i = r$, we have

$$ch_1(E).P = \xi_{P,\star}\,(ch_1(E \mid_P)) = ch_1\left(\bigoplus_{i=1}^{k} \mathcal{O}(m_i)\right) = \sum_{i=1}^{k} m_i r_i, \text{ then}$$

$$\Delta^{Vb}(E',P) = \Delta^{Vb}(E,P) - \frac{r_1}{r}\sum_{i=1}^{k} m_i r_i - \frac{r_1^2}{2r} + m_1 r_1 + \frac{r_1}{2} = \Delta(E) - \frac{1}{r}\mathcal{A}, \text{ where}$$

$$\mathcal{A} = \sum_{i=2}^{k} m_i r_i r_1 + m_1 r_1^2 + \frac{1}{2} r_1^2 - (m_1 + \frac{1}{2}) r_1 r \quad for \quad r = r_1 + ... + r_k$$

$$= \sum_{i=2}^{k} m_i r_i r_1 + m_1 r_1^2 + \frac{1}{2} r_1^2 - (m_1 + \frac{1}{2}) r_1^2 - \sum_{i=2}^{k} (m_1 + \frac{1}{2}) r_1 r_i, \text{ then}$$

$$\mathcal{A} = \sum_{i=2}^{k} (m_i - m_1 - \frac{1}{2}) r_1 r_i \qquad \text{where } m_i > m_1 + 1$$

Note that, with our hypothesis that $E\mid_P$ is not pure, we have $m_i \geq m_1 + 1$ so $A > 0$.

Proposition 5.13. *If $E|_P$ is not pure, then let E' be the elementary transformation considered above. The local invariant satisfies*

$$\Delta^{Vb}_{loc}(E',P) = \Delta^{Vb}_{loc}(E,P) - \frac{1}{r}\sum_{i=2}^{k} (m_i - m_1 - \frac{1}{2}) r_1 r_i \quad where \quad m_i > m_1 + 1.$$

In particular, $\quad \Delta^{Vb}_{loc}(E',P) < \Delta^{Vb}_{loc}(E,P)$.

If E' is pure then $\Delta^{Vb}_{loc}(E',P) = 0$, if not we can continue by applying the elementary transformation process to E' and so on, until the result is pure. The resulting theorem can be viewed as a local analogue of the Bogomolov-Gieseker inequality.

Theorem 5.14. *If E is a vector bundle on X and $P \cong \mathbf{P}^1 \subset X$ is the exceptional divisor of blowing up a smooth point $\check{P} \in \check{X}$, then $\Delta^{Vb}_{loc}(E,P) \geq 0$, and $\Delta^{Vb}_{loc}(E,P) = 0$ if and only if $E \cong \mu^*(\check{E})$ is the pullback of a bundle from $\check{X}$.*

The invariant $\Delta^{Vb}_{loc}(X,P)$ also provides a bound for $m_i - m_1$.

Corollary 5.15. *If $E\mid_P = \bigoplus_{i=1}^{k} \mathcal{O}(m_i)^{r_i}$, where $m_1 < m_2 < ... < m_k$. Then*

$$\Delta^{Vb}_{loc}(E',P) \geq 0; \quad \Delta^{Vb}_{loc}(E,P) \geq \frac{1}{r}\sum_{i=2}^{k} (m_i - m_1 - \frac{1}{2}) r_1 r_i.$$

6. Modification of filtrations due to elementary transformations

Given two bundles E and F such that $E|_U \cong F|_U$, then F may be obtained from E by a sequence of elementary transformations. We therefore analyse what happens to the filtrations along the divisor components D_i different from exceptional divisors P_u, in the case of an elementary transformation.

Suppose E' is obtained from E by an elementary transformation.

We have bundles $Gr(D_i, k; E)$ and $Gr(D_i, k; E')$ over D_i. In order to follow the modification of the formula for Δ we need to consider this change.

For the bundle E we have a filtration by full flags $F^i_k \subset E|_{D_i}$. Suppose $E|_{D_0} \to Q$ is a quotient (locally free on D_0) and let E' be the elementary transformation fitting into the exact sequence

$$0 \to E' \to E \to Q \to 0.$$

Lemma 6.1. *Suppose $i \neq 0$ so $D_i \cap D_0$ is transverse. Tensoring this exact sequence with $\mathcal{O}_{D_i}$ yields an exact sequence*

$$0 \to E'|_{D_i} \to E|_{D_i} \to Q|_{D_i} \to 0.$$

Proof. In fact we get a long exact sequence

$$Tor^1_{\mathcal{O}_X}(\mathcal{O}_{D_i}, Q) \to E' \otimes \mathcal{O}_{D_i} \to E \otimes \mathcal{O}_{D_i} \to Q \otimes \mathcal{O}_{D_i} \to 0,$$

but the facts that Q is locally free on D_0 and D_i is transverse to D_0 imply that $Tor^1_{\mathcal{O}_X}(\mathcal{O}_{D_i}, Q) = 0$. $\square$

This lemma says that $E'|_{D_i}$ is an elementary transformation of $E|_{D_i}$. Notice that since D_i is a curve, $Q|_{D_i}$ is a skyscraper sheaf.

Define $F'^i_k := F^i_k \cap (E'|_{D_i})$. It is a subsheaf of $(E'|_{D_i})$. Furthermore it is saturated, that is to say the quotient is torsion-free. To show this, suppose s is a section of $(E'|_{D_i})$ which is contained in F'^i_k over an open set. Then it may be seen as a section of $E|_{D_i}$ which is contained in F^i_k over an open set, but F^i_k is saturated so the section is contained in F^i_k. Hence by definition the section is contained in F'^i_k.

Thus, we have defined a filtration F'^i_k by sub-vector bundles. The same argument says that F'^i_k is saturated in F'^i_{k+1}, so the quotients $Gr(D_i, k; E') = F'^i_{k+1}/F'^i_k$ are locally free; since they are line bundles over the open set, they are line bundles on D_i.

Consider the morphism, induced by $(E'|_{D_i}) \to (E|_{D_i})$:

$$F'^i_{k+1}/F'^i_k \to F^i_{k+1}/F^i_k.$$

By the definition of F'^i_k it is seen that this morphism is an injection of sheaves. Consider the cokernel. If s is a section of F^i_{k+1}/F^i_k and if z is a local coordinate on D_i such that $z = 0$ defines the intersection point $D_0 \cap D_i$, then we claim that zs must be in the image of F'^i_{k+1}/F'^i_k. Lift s to a section also denoted s of F^i_{k+1}. Then, thought of as a section of $E|_{D_i}$, notice that zs projects to 0 in $Q|_{D_i}$. This may be seen by further extending to a section of E and extending z to a coordinate function defining D_0; noting that Q is supported scheme-theoretically on D_0 so zs projects to 0 in Q.

From the exact sequence, we conclude that zs is in the image of F'^i_{k+1}/F'^i_k. Hence, there are two cases:

(1) the map $F'^i_{k+1}/F'^i_k \to F^i_{k+1}/F^i_k$ is an isomorphism; or

(2) we have $F'^i_{k+1}/F'^i_k = F^i_{k+1}/F^i_k \otimes_{\mathcal{O}_{D_i}} \mathcal{O}_{D_i}(-D_0 \cap D_i)$.

In the first case (1),

$$c_1^{D_i}(Gr(D_i,k;E')) = c_1^{D_i}(Gr(D_i,k;E)).$$

In the second case (2),

$$c_1^{D_i}(Gr(D_i,k;E')) = c_1^{D_i}(Gr(D_i,k;E)) - [D_0 \cap D_i].$$

Applying $(\xi_i)_*$ gives the following proposition.

Proposition 6.2. *Suppose E' is an elementary transformation of E. Then there exist a unique invariant $\deg_{loc}$ that satisfy the following properties:*

$$\deg_{loc}\left(D_j,k;\check{E}\right) := 0,$$
$$\deg_{loc}\left(D_j,k;\check{E}(m.P_i)\right) := m,$$

and for divisor components D_i intersecting D_0 transversally, the change in Chern class of the associated-graded pieces is

$$\deg_{loc}\left(Gr(D_j,k;E'),P_i\right) := \deg_{loc}\left(Gr(D_j,k;E),P_i\right) - \tau(E,E';k)$$

where $\tau(E,E';k) = 0$ or 1 in cases (1) or (2) respectively.

Definition 6.3. *Let S and $\check{S}$ are two sheaves of finite length with support at points P_i. Suppose We have bundles $Gr(D_j,k,E)$ and $Gr(D_j,k,\check{E})$ over D_j respectively $\check{D}_j$. Let F be the intersection of subsheaves $Gr(D_j,k,E)$ and $Gr(D_j,k,\check{E})$. Define lg to be the length, and let $lg(S,P_i)$ be the length of the part supported set-theoretically at P_i. Thus*

$$lg(S) = \sum_i lg(S,P_i)$$

and similarly for $\check{S}$. Consider the sequences:

$$F \longrightarrow Gr(D_j,k,E) \longrightarrow S \longrightarrow 0$$
$$F \longrightarrow Gr(D_j,k,\check{E}) \longrightarrow \check{S} \longrightarrow 0.$$

Define

$$\deg_{loc}\left(Gr(D_j,k;E,P_i)\right) := lg(S,P_i) - lg(\check{S},P_i).$$

Then

$$deg\left(Gr(D_j,k,E)\right) = deg(F) + lg(S)$$
$$deg\left(Gr(D_j,k,\check{E})\right) = deg(F) + lg(\check{S}),$$

therefore

$$lg(S) - lg(\check{S}) = \sum_{P_i}\left[lg(S,P_i) - lg(\check{S},P_i)\right] = \sum_{P_i} deg_{loc}\left(Gr(D_j,k;E,P_i)\right).$$

Suppose $i \neq 0$, if D_j are non-exceptional divisor. Then

$$deg\left(Gr(D_j,k;E\right) = deg\left(Gr(\check{D}_j,k;\check{E}\right) + \sum_{P_i} deg_{loc}\left(Gr(D_j,k,E),P_i)\right).$$

This completes the proof of the proposition.

7. The local parabolic invariant

Let E be a bundle, with $\beta(D_0, k) = 0, \forall k$. Then we would like to define the terms in the following equation:

$$\Delta^{Par}(E) = \Delta^{Par}(\check{E}) + \sum_{P_i} \Delta^{Par}_{loc}(E, P_i).$$

Assume that $\check{D}$ is a union of smooth divisors meeting in some multiple points. The divisor D is obtained by blowing up the points $\check{P}_u$ of multiplicity ≥ 3. Let

$$\varphi : X \to \check{X}$$

be the birational transformation. We use the previous formula to break down $\Delta^{Par}(E)$ into a global contribution which depends only on $\check{E}$, plus a sum of local contributions depending on the choice of extension of the parabolic structure across P_u.

Let $\check{\mathcal{S}}$ denote the set of divisor components in $\check{D}$ (before blowing-up) and define the global term $\Delta^{Par}(\check{E})$ by the formula

$$\begin{aligned}
\Delta^{Par}(\check{E}) := \Delta^{Vb}(\check{E}) \\
&+ \sum_{i \in \check{\mathcal{S}}} \sum_{k=1}^{r} \beta(\check{D}_i, k).deg\big(Gr(\check{D}_i, k)\big) \\
&- \frac{1}{2} \sum_{i \in \mathcal{S}} \sum_{k=1}^{r} \beta(\check{D}_i, k)^2.[\check{D}_i]^2 \\
&- \frac{1}{2} \sum_{i \neq j} \sum_{k=1}^{r} \sum_{y \in Irr(\check{D}_i \cap \check{D}_j)} \beta(\check{D}_i, k).\beta(\check{D}_j, \sigma(y, i, j)(k)).[y].
\end{aligned}$$

This formula imitates the formula for Δ^{Par} by considering only pairwise intersections of divisor components even though several different pairwise intersections could occur at the same point. Recall that $[D_i]^2 = [\check{D}_i]^2 - m$ where m is the number of points on $\check{D}_i$ which are blown up to pass to D_i.

To define the local terms, suppose at least one of the divisors, say $D_0 = P$, is the exceptional locus for a birational transformation blowing up the point $\check{P}$. We define a local contribution $\Delta^{Par}_{loc}(E, P)$ to Δ^{Par} by isolating the local contributions in the previous formula.

Notice first of all that for any D_i meeting P transversally, we have defined above $deg_{loc}(Gr(E; D_i, k), P)$, the local contribution at P, in such a way that

$$deg(Gr(E; D_i, k)) = deg\big(Gr(\check{E}; D_i, k)\big) + \sum_{P_u} deg_{loc}(Gr(E; D_i, k), P_u)$$

where the sum is over the exceptional divisors P_u meeting D_i, which correspond to the points $\check{P}_u \in \check{D}_i$ which are blown up.

Let $\mathcal{S}(P)$ denote the set of divisor components which meet P but not including $P = D_0$ itself. Define

$$\begin{aligned}
\Delta^{Par}_{loc}(E, P) := \Delta^{Vb}_{loc}(E, P)& \\
&+ \sum_{k=1}^{r} \beta(P, k).deg(Gr(E; P, k)) \\
&+ \sum_{i\in\mathcal{S}(P)} \sum_{k=1}^{r} \beta(D_i, k).deg_{loc}(Gr(E; D_i, k), P) \\
&+ \frac{1}{2} \sum_{k=1}^{r} \beta(P, k)^2 \\
&+ \frac{1}{2} \sum_{i\in\mathcal{S}(P)} \sum_{k=1}^{r} \beta(D_i, k)^2 \\
&- \sum_{i\in\mathcal{S}(P)} \sum_{k=1}^{r} \beta(D_i, k).\beta(P, \sigma(i, P)(k)).[y] \\
&+ \frac{1}{2} \sum_{i\neq j, \check{P}\in\check{D}_i\cap\check{D}_j} \sum_{k=1}^{r} \beta(\check{D}_i, k).\beta(\check{D}_j, \sigma(\check{P}, i, j)(k)).[\check{P}]
\end{aligned}$$

In the next to last term, $\sigma(i, P) := \sigma(y, i, v)$ where $P = D_v$ and y is the unique intersection point of $P = D_v$ and D_i. The factor of $1/2$ disappears because we are implicitly choosing an ordering of the indices $i, j = 0$ which occur here. The last term is put in to cancel with the corresponding term in the global expression for $\check{E}$ above, and $[\check{P}]$ designates any lifting of the point $\check{P}$ to a point on P.

Theorem 7.1. *With the above definitions, we have*

$$\Delta^{Par}(E) = \Delta^{Par}(\check{E}) + \sum_{P_u} \Delta^{Par}_{loc}(E, P_u),$$

where the sum is over the exceptional divisors.

Proof. This follows by comparing the above definitions with the formula of Proposition 3.12. □

Let $\varphi^*\check{E}$ denote the parabolic bundle on X given by using the trivial extension φ^*E as underlying vector bundle, and setting $\beta(P_u,k) := 0$ for all exceptional divisor components P_u. Note that $\Delta^{Vb}_{loc}(\varphi^*E,P)=0$. Then

$$\Delta^{Par}_{loc}(\varphi^*\check{E},P) = \frac{1}{2}\sum_{i\in\mathcal{S}(P)}\sum_{k=1}^{r}\beta(\check{D}_i,k)^2$$

$$+\frac{1}{2}\sum_{i\neq j,\check{P}\in\check{D}_i\cap\check{D}_j}\sum_{k=1}^{r}\beta(\check{D}_i,k).\beta(\check{D}_j,\sigma(\check{P},i,j)(k)).[\check{P}],$$

and

$$\Delta^{Par}_{loc}(E,P)-\Delta^{Par}_{loc}(\varphi^*\check{E},P)=\Delta^{Vb}_{loc}(E,P)$$

$$+\sum_{k=1}^{r}\beta(P,k).deg(Gr(E;P,k))$$

$$+\sum_{i\in\mathcal{S}(P)}\sum_{k=1}^{r}\beta(D_i,k).deg_{loc}(Gr(E;D_i,k),P)$$

$$+\frac{1}{2}\sum_{k=1}^{r}\beta(P,k)^2$$

$$-\sum_{i\in\mathcal{S}(P)}\sum_{k=1}^{r}\beta(D_i,k).\beta(P,\sigma(i,P)(k)).[y].$$

A different local-global decomposition may be obtained by noting that

$$\Delta^{Par}(E)=\Delta^{Par}(\varphi^*\check{E})+\sum_u\left(\Delta^{Par}_{loc}(E,P_u)-\Delta^{Par}_{loc}(\varphi^*\check{E},P_u)\right)$$

with the local terms $(\Delta^{Par}_{loc}(E,P_u)-\Delta^{Par}_{loc}(\varphi^*\check{E},P_u))$ being given by the previous formula.

8. Normalization via standard elementary transformations

There is another modification of parabolic structures due to elementary transformations. This may also be viewed as a shift of the parabolic structures in the viewpoint of a collection of sheaves. If $E_{\cdot}=\{E_{\alpha_1,\ldots,\alpha_n}\}$ is a parabolic sheaf, then we can shift the filtration at the i-th place defined by

$$(C^i_\theta E)_{\alpha_1,\ldots,\alpha_n} := E_{\alpha_1,\ldots,\alpha_i-\theta,\ldots,\alpha_n}.$$

This may also be viewed as tensoring with a parabolic line bundle

$$C^i E = E \otimes \mathcal{O}(\theta D_i).$$

The weights of the parabolic structure $C^i_\theta E$ along D_i are of the form $\alpha_i + \theta$ for α_i weights of E.

In the point of view of a vector bundle with filtration, it may correspond to doing an elementary transformation. Suppose $0 < \theta < 1$. Then

$$(C^i_\theta E)_0 = E_{0,\ldots,0,-\theta,0,\ldots,0}$$

and we have an exact sequence

$$0 \to (C^i_\theta E)_0 \to E_0 \to (E_0/F^i_{-\theta}E_0) \to 0.$$

Therefore $(C^i_\theta E)_0$ is obtained by elementary transformation of E_0 along one of the elements of the parabolic filtration on the divisor D_i.

This is specially useful when the rank is 2. Suppose $rk(E) = 2$. There is a single choice for the elementary transformation. If the weights of E at D_i are $\alpha_i^{tot} - \beta_i$ and $\alpha_i^{tot} + \beta_i$ then the weights of the elementary transformation will be $\theta + \alpha_i^{tot} + \beta_i - 1$ and $\theta + \alpha_i^{tot} - \beta_i$. The shift θ should be chosen so that these lie in $(-1, 0]$. The new weights may be written as

$$(\tilde{\alpha}_i^{tot} - \tilde{\beta}_i, \quad \tilde{\alpha}_i^{tot} + \tilde{\beta}_i$$

with

$$\tilde{\alpha}_i^{tot} := (\theta + \alpha_i^{tot} - \frac{1}{2})$$

which is the new average value, and

$$\tilde{\beta}_i := \frac{1}{2} - \beta_i.$$

Corollary 8.1. *In the case $rk(E) = 2$, by replacing E with its shift $C^i_\theta E$ if necessary, we may assume that*

$$0 \le \beta_i \le \frac{1}{4}.$$

Proof. If $\beta_i > \dfrac{1}{4}$ then do the shift which corresponds to an elementary transformation; for the new parabolic structure $\tilde{\beta}_i = \dfrac{1}{2} - \beta_i$ and $0 \le \tilde{\beta}_i \le \dfrac{1}{4}$. □

9. The rank two case

In order to simplify the further constructions and computations, we now restrict to the case when E has rank 2. The parabolic structures along D_i are rank one subbundles $F^i \subset E|_{D_i}$. The associated graded pieces are $Gr(D_i, 1) = F^i$ and $Gr(D_i, 2) = E|_{D_i}/F^i$. The normalized weights may be written as

$$\beta(D_i, 1) = -\beta_i, \quad \beta(D_i, 2) = \beta_i$$

with $0 \leq \beta_i < \frac{1}{2}$, and by Corollary 8.1 we may furthermore suppose $0 \leq \beta_i \leq \frac{1}{4}$.

Define $\deg^{\delta}(E_{D_i}, F^i) := \big(\deg(E_{D_i}/F^i) - \deg(F^i)\big)$. This has a local version as discussed in Definition 6.3,

$$\deg^{\delta}_{loc}(E_{D_i}, F^i, P) := \big(\deg_{loc}(E_{D_i}/F^i, P) - \deg_{loc}(F^i, P)\big)$$

whenever D_i meets P transversally.

The main formula may now be rewritten:

Proposition 9.1. $\Delta^{Par}(E) = \Delta^{Vb}(E)$

$$\begin{aligned}
&+ \sum_{i\in\mathcal{S}} \beta_i \deg^{\delta}(E_{D_i}, F^i) \\
&- \sum_{i\in\mathcal{S}} \beta_i^2 . [D_i]^2 \\
&- \sum_{i\neq j} \sum_{y\in Irr(D_i\cap D_j)} \tau(y,i,j)\beta_i\beta_j.[y]
\end{aligned}$$

where $\tau(y,i,j) = 1$ if $F^i(y) = F^j(y)$ and $\tau(y,i,j) = -1$ if $F^i(y) \neq F^j(y)$ as subspaces of $E(y)$.

Similarly for the local parabolic invariants, denoting $P = D_0$ we have

$$\Delta^{Par}_{loc}(E,P) - \Delta^{Par}_{loc}(\varphi^*\check{E}, P) = \Delta^{Vb}_{loc}(E,P)$$

$$\begin{aligned}
&+ \beta_0. \deg^{\delta}(E_{D_0}, F^0) \\
&+ \sum_{i\in\mathcal{S}(P)} \beta_i. \deg^{\delta}_{loc}(E_{D_i}, F^i, P) \\
&+ \beta_0^2 \\
&- 2 \sum_{i\in\mathcal{S}(P)} \tau(i,P)\beta_i\beta_0.[y].
\end{aligned}$$

Example 9.2. *Let E be a non pure rank two bundle, we have $E\mid_P = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$, then* $\Delta^{Vb}_{loc}(E,P) \geq \frac{1}{2}(m_2 - m_1 - \frac{1}{2})$.

Example 9.3. *Let E be a rank 2 bundle with $E|_P = \mathcal{O} \oplus \mathcal{O}(1)$. The reduction by elementary transformation is $E|_P = \mathcal{O} \oplus \mathcal{O}(1) \rightsquigarrow E'$. We get an exact sequence*

$$0 \longrightarrow \mathcal{O}(1) \longrightarrow E'|_P \longrightarrow \mathcal{O}(1) \longrightarrow 0$$

then E' is pure, $E' = \mu^{\star}\check{E}(-P)$.

Suppose we start with the bundle E, by doing the sequence of elementary transformation we get $\check{E}(m.P_i)$. Number the sequence in opposite direction, we get a sequence of bundles of the form:

$$\check{E}(m.P_i) = E(0),\ E(1),\ E(2),\ ...,\ E(g) = E$$

where g is the number of steps, and $E(j-1) = (E(j))'$ for $j = 1, ..., g$. we recall that if $E|_P \cong \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$ with $m_1 \leq m_2$ then $\mu(E) = m_2 - m_1$. Also $\mu(E) = 0 \Longrightarrow E = \check{E}(m_i P_i)$. Furthermore if $m_1 < m_2$ then $\mu(E') < \mu(E)$. We see that

$$0 = \mu(E_0) < \mu(E_1) < \mu(E_2) < ... < \mu(E_{g-1}).$$

To calculate $\Delta(E, P)_{loc}$ we use the proposition 5.13 applied to each $E(j)$:

$$\Delta^{Vb}_{loc}(E(j)', P_0) = \Delta^{Vb}_{loc}(E(j), P_0) - \frac{1}{2}\mathcal{A},$$

where

$$\mathcal{A} = \sum_{i=2}^{k}(m_2 - m_1 - \frac{1}{2})r_1 r_i = m_2(E(j)) - m_1(E(j)) - \frac{1}{2} = \mu(E(j)) - \frac{1}{2}.$$

Therefore

$$\Delta^{Vb}_{loc}(E(j-1)) = \Delta^{Vb}_{loc}(E(j)) - \frac{1}{2}(\mu(E(j)) - \frac{1}{2}),$$

and putting them all together,

$$\Delta^{Vb}_{loc}(E(0)) = 0;\ \ \Delta^{Vb}_{loc}(E(g)) = \frac{1}{2}\sum_{i=1}^{g}(\mu(E(j)) - \frac{1}{2}).$$

We have

$$\mu(E(j-1)) < \mu((E(j)), \tag{1}$$

so $\mu(E(j)) \geq j$.

Now we divide the work in two parts, first term $\mu(E(g)) - \frac{1}{2}$, then the sum of the others. For each $\mu(E(j))$ is at least one greater than the previous one, this gives that the sum of the other terms is at least equal to $(1+2+3+...+(g-1)) - \frac{(g-1)}{2}$. Then

$$\Delta^{Vb}_{loc}(E(g)) \geq \frac{1}{2}(1+2+3+...+(g-1)) - \frac{g}{4} + \frac{1}{2}\mu(E(g))$$

$$= \frac{g(g-1)}{4} - \frac{g}{4} + \frac{1}{2}\mu(E).$$

We have therefore proven the following:

Proposition 9.4. *If E is a bundle which is brought to pure form in $g \geq 1$ steps of elementary transformation, and $\mu(E) = m_2(E|_P) - m_1(E|_P)$, then we have the lower bound*

$$\Delta_{loc}^{Vb}(E) = \Delta_{loc}^{Vb}(E(g)) \geq \frac{g^2 - 2g}{4} + \frac{1}{2}\mu(E).$$

We have for each $1 \leq k \leq g$,

$$\left|\deg_{loc}^{\delta}(E(k), F^i, P) - \deg_{loc}^{\delta}(E(k-1), F^i, P)\right| \leq 1,$$

but also $E(0) = \varphi^*(\check{E})$ and $\deg_{loc}^{\delta}(E(0), F^i, P) = 0$, so

$$\left|\deg_{loc}^{\delta}(E(g), F^i, P)\right| \leq g.$$

Also along P we have $E_P = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$ with $m_1 \leq m_2$. For any subbundle $F^0 \subset E_P$ we have $\deg(F^0) \leq m_2$ and $\deg(E_P/F^0) \geq m_1$ so

$$\deg^{\delta}(E_P, F^0) \geq m_1 - m_2 = -\mu(E).$$

Then

$$\begin{aligned}
\Delta_{loc}^{Par}(E, P) - \Delta_{loc}^{Par}(\varphi^*\check{E}, P) = \Delta_{loc}^{Vb}(E, P)& \\
&+ \beta_0 . \deg^{\delta}(E_{D_0}, F^0) \\
&+ \sum_{i \in \mathcal{S}(P)} \beta_i . \deg_{loc}^{\delta}(E_{D_i}, F^i, P) \\
&+ \beta_0^2 \\
&- 2 \sum_{i \in \mathcal{S}(P)} \tau(i, P)\beta_i\beta_0 . [y] \\
&\geq \frac{g^2 - 2g}{4} \quad + \quad \frac{1}{2}\mu(E) \\
&- \beta_0\mu(E) \\
&- \sum_{i \in \mathcal{S}(P)} \beta_i g \\
&+ \beta_0^2 \\
&- 2 \sum_{i \in \mathcal{S}(P)} \beta_i\beta_0 .
\end{aligned}$$

But we know that $\mid \beta_i \mid \leq \dfrac{1}{2}$. Then we get the following theorem.

Theorem 9.5.
$$\Delta^{Par}_{loc}(E;P) \geq \Delta^{Par}_{loc}(\varphi^*\check{E},P) + \frac{g^2-2g}{4} - \frac{g+1}{2}.\kappa.$$
Where $\kappa = \#\mathcal{S}(P)$ is the number of divisors of D_i meeting P.

Theorem 9.6. *If $\check{E}$ is a vector bundle of rank 2 on $\check{X}$ with parabolic structures on the components $\check{D}_i$, then on X obtained by blowing up the multiple points of $\check{D}$, the parabolic invariant $\Delta^{Par}_{loc}(E,P)$ attains a minimum for some extension of the bundle E and some parabolic structures on the exceptional loci.*

Proof. From the above theorem, the number of elementary transformations g needed to get to any E with $\Delta^{Par}_{loc}(E;P) \leq \Delta^{Par}_{loc}(\varphi^*\check{E},P)$, is bounded. Furthermore the number of numerical possibilities for the degrees $\deg^{\delta}_{loc}(E_{D_i},F^i,P)$ and $\deg^{\delta}(E_P,F^0,P)$ leading to such a minimum, is finite. The parabolic weight β_0 may be chosen to lie in the closed interval $[0,\frac{1}{4}]$, so the set of possible numerical values lies in a compact subset; hence a minimum is attained. □

Denote the parabolic extension which achieves the minimum by E^{min}. There might be several possibilities, although we conjecture that usually it is unique. Thus
$$\Delta^{Par}_{loc}(E^{min},P) = \min_E\left(\Delta^{Par}_{loc}(E,P)\right).$$
With the minimum taken over all parabolic extensions E of $\check{E}|_{\check{U}}$ across the exceptional divisor P.

The minimal E^{min} exists at each exceptional divisor and they fit together to give a global parabolic bundle. Define

$$\Delta^{Par}_{min}(\check{E}) := \Delta^{Par}(E^{min})$$
$$= \Delta^{Par}(\check{E}) + \sum_{P_u} \Delta^{Par}_{loc}(E^{min},P_u).$$

9.1. **Panov differentiation.** D. Panov in his thesis [Pa] used the idea of differentiation with respect to the parabolic weight. A version of this technique allows us to gain more precise information on the minimum.

Lemma 9.7. *Let $E = E^{min}$ be the parabolic bundle extending $\check{E}|_{\check{U}}$ which achieves the minimum value $\Delta^{Par}_{loc}(E^{min},P)$. By making an elementary transformation we may assume $0 \leq \beta_0 \leq \frac{1}{4}$. Denote also by E the underlying vector bundle. Then for any subbundle $F' \subset E|_P$ we have*
$$\deg(E|_P/F') - \deg(F') \geq -\kappa.$$

Thus if $E|_P = \mathcal{O}_P(m_1) \oplus \mathcal{O}_P(m_2)$ *then*

$$|m_2 - m_1| \le \kappa.$$

Where $\kappa = \#\mathcal{S}(P)$ *is the number of divisors of* D_i *meeting* P.

Proof. We show that $\deg(E|_P/F') - \deg(F') \ge -\kappa - \frac{1}{4}$ which implies the stated inequality since the left side and κ are integers. Let $F = F^0 \subset E|_P$ be the subbundle corresponding to the parabolic structure E^{min}. Consider two cases:
(i) if F^0 is the destabilizing bundle of $E|_P$ and $\beta_0 > 0$; or
(ii) if F^0 is not the destabilizing bundle of $E|_P$, or else $\beta_0 = 0$.

In case (i) note that β_0 may be allowed to range in the full interval $[0, \frac{1}{2})$ so the invariant $\Delta^{Par}_{loc}(E, P)$ is a local minimum considered as a function of $\beta_0 \in (0, \frac{1}{2})$. Then

$$\frac{d}{d\beta_0}\Delta^{Par}_{loc}(E, P) = 0.$$

This gives the formula

$$\deg^\delta(E_{D_0}, F^0) = 2\ \beta_0 +\ 2 \sum_{i \in \mathcal{S}(P)} \tau(F^i, F^0)\beta_i, \text{ so}$$

$$\deg^\delta(E_{D_0}, F^0) \ge -\frac{\kappa}{2}.$$

Since F^0 is the destabilizing bundle it implies that

$$\deg^\delta(E_{D_0}, F') \ge -\frac{\kappa}{2}$$

for any other subbundle F' also, which is stronger than the desired inequality in this case.

In case (ii) we have $\beta_0. \deg^\delta(E_{D_0}, F^0) \ge 0$ because in the contrary case that would imply that F^0 is the destabilizing subbundle. Suppose $F' \subset E|_P$ is a possibly different subbundle such that

$$\deg(E|_P/F') - \deg(F') < -\frac{1}{4}(1 + 4\kappa).$$

Then make a new parabolic structure E' using F' instead of F, with parabolic weight $\beta_0' = \frac{1}{4}$. We have

$$\Delta^{Par}_{loc}(E', P) - \Delta^{Par}_{loc}(E, P) =$$

$$\frac{1}{4}\deg^\delta(E_{D_0}, F')$$

$$- \beta_0 . \deg^\delta(E_{D_0}, F^0)$$

$$+\frac{1}{16}$$

$$- \beta_0^2$$

$$+ 2 \sum_{i \in \mathcal{S}(P)} \tau(F, F^i)\beta_i\beta_0.[y]$$

$$- \frac{1}{2} \sum_{i \in \mathcal{S}(P)} \tau(F', F^i)\beta_i.[y]$$

$$\leq \frac{1}{4} \deg^\delta(E_{D_0}, F') + \frac{1}{16}(1 + 4\kappa)$$

$$< 0.$$

This contradicts minimality of E^{min}, which shows the desired inequality. □

Corollary 9.8. *In the case of* 3 *divisor components* $\kappa = 3$ *and the minimal extension* E^{min} *satisfies* $|m_2 - m_1| \leq 3$. *It is connected to* $\varphi^*(\check{E})$ *by at most three elementary transformations.*

This should permit an explicit description of all possible cases for $\kappa = 3$.

9.2. **The Bogomolov-Gieseker inequality.** Suppose $C \subset \check{X}$ is an ample curve meeting $\check{D}$ transversally. Then $\check{E}|_C$ is a parabolic bundle on C.

Proposition 9.9. *Suppose* $\check{E}|_C$ *is a stable parabolic bundle. Then for any extension* E *to a parabolic bundle over* X, *there exists an ample divisor* H *on* X *such that* E *is* H*-stable. Hence* $\Delta^{Par}(E) \geq 0$. *In particular* $\Delta^{Par}(E^{min}) \geq 0$. *If* $\check{E}$ *comes from an irreducible unitary representation of* $\pi_1(\check{X} - \check{D})$ *then the parabolic extension on* X *corresponding to the same unitary representation must be some choice of* E^{min}.

Proof. Fix an ample divisor H'. Then any divisor of the form $H = nC + H'$ is ample on X, and for n sufficiently large E will be H-stable. The Bogomolov-Gieseker inequality for parabolic bundles says that $\Delta^{Par}(E) \geq 0$ with equality if and only if E comes from a unitary representation. However, $\Delta^{Par}(E) \geq \Delta^{Par}(E^{min}) \geq 0$ and if E comes from a unitary representation then $\Delta^{Par}(E) = \Delta^{Par}(E^{min}) = 0$. It follows in this case that E is one of the choices of E^{min}. □

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Laboratoire J. A. Dieudonné, UMR 6621 Université Nice Sophia-Antipolis, 06108 Nice, Cedex 2, France.
E-mail address: **taher@math.unice.fr, chaditaher@hotmail.fr**